

A Note on the Minimum Number of Edges in Hypergraphs with Property O

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Abstract

An oriented k -uniform hypergraph has Property O, if for every linear order of the vertex set, there is some edge oriented consistently with the linear order.

In [3] Duffus, Kay and Rödl investigate the minimum number $f(k)$ of edges a k -uniform hypergraph having *Property O* can have. They prove $k! \leq f(k) \leq (k^2 \ln k)k!$, where the upper bound holds for sufficiently large k . In this note we improve the upper bound by a factor of $k \ln k$ showing $f(k) \leq (\lfloor \frac{k}{2} \rfloor + 1)k! - \lfloor \frac{k}{2} \rfloor (k-1)!$ for every $k \geq 3$.

Furthermore they introduce the minimum number $n(k)$ of vertices a k -uniform hypergraph having Property O can have. For $k = 3$ they show $6 \leq n(3) \leq 9$ and ask for the precise value of $n(3)$. We show that $n(3) = 6$.

1 Introduction

A *hypergraph* is a pair $H = (V, E)$, where V is a finite set whose elements are called *vertices* and E is a family of subsets of V , called *edges*. It is *k -uniform* if every edge contains precisely k vertices. When studying certain hypergraph properties, one is often interested in a question of the following form: what is the minimum possible number of edges a k -uniform hypergraph can have, such that it does/doesn't have the property of interest. Arguably the most famous example is *property B*, first introduced by Bernstein in 1908 [2] (see [1] for a good overview), asking for the minimum possible number of edges a k -uniform hypergraph can have such that its vertex set is not properly two-colourable.

In a recent paper [3], Duffus, Kay and Rödl introduce the following property, called *Property O*. Fix an integer $k \geq 2$ and some finite set V . An *ordered k -set* is a k -tuple $\bar{e} = (x_1, \dots, x_k)$ of distinct elements of V . We write e to denote the underlying k -set of \bar{e} . An *oriented k -uniform*

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hypergraph, or *oriented k -graph*, is a pair $\mathcal{H} = (V, \mathcal{E})$, where $\mathcal{E} \subset V^k$ is a family of ordered k -sets with no two k -tuples forming the same k -element set. In the case that \mathcal{E} contains an ordered k -set for every k -subset of V , we call \mathcal{H} a *k -tournament*.

Given a linear order $<$ on V , we say that an ordered k -set $\bar{e} = (x_1, x_2, \dots, x_k)$ is *consistent* with $<$, if $x_1 < x_2 < \dots < x_k$. For convenience, we shall then simply say that \bar{e} is $<$ -consistent.

Definition 1.1. Let $k \geq 2$ and let $\mathcal{H} = (V, \mathcal{E})$ be an oriented k -graph. We say that \mathcal{H} has the ordering property or Property O, if for every linear order $<$ on V , there exists $\bar{e} \in \mathcal{E}$ that is consistent with $<$. Furthermore, let

$$f(k) := \min\{|\mathcal{E}| : \text{there exists an oriented } k\text{-graph } \mathcal{H} = (V, \mathcal{E}) \text{ having Property O}\}.$$

In words, $f(k)$ is the minimum number of edges in an oriented k -graph having Property O. It is easy to check that $f(2) = 3$ and an example for the upper bound is a cyclically ordered triangle. Apart from this trivial case, the following is known about $f(k)$:

Theorem 1.2 (Duffus-Kay-Rödl, [3]). *The function $f(k)$ satisfies $k! \leq f(k) \leq (k^2 \ln(k))k!$ where the lower bounds holds for all k and the upper bound for k sufficiently large.*

Their proof of the upper bound is probabilistic: they showed that a randomly chosen k -tournament on n vertices with $(k^2 \ln(k)k!)$ edges has Property O with positive probability (for suitably chosen n and k sufficiently large). Furthermore they showed that almost all k -tournaments with $(1 - o(1))\sqrt{k} \cdot k!$ edges don't have Property O.

The aim of this note is to prove the following.

Theorem 1.3. *Let $k \geq 3$. Then there exists an oriented k -graph with $(\lfloor \frac{k}{2} \rfloor + 1)k! - \lfloor \frac{k}{2} \rfloor(k-1)!$ edges with Property O. Hence $f(k) \leq (\lfloor \frac{k}{2} \rfloor + 1)k! - \lfloor \frac{k}{2} \rfloor(k-1)!$.*

Note that, in contrast to Theorem 1.2, our upper bound holds for all $k \geq 3$. However, the question whether $f(k)$ is bounded away from $k!$ remains open.

Unfortunately we are not able to improve the (trivial) lower bound for general k , namely $f(k) \geq k!$. We will include its proof here for the convenience of the reader. Suppose that $\mathcal{H} = ([n], \mathcal{E})$ is an oriented k -graph that has Property O. Then every $\bar{e} \in \mathcal{E}$ is consistent with

$$\binom{n}{k}(n-k)! = \frac{n!}{k!}$$

orders on $[n]$. Since \mathcal{H} has Property O, we must have

$$|\mathcal{E}| \cdot \frac{n!}{k!} \geq n!$$

and hence $f(k) \geq k!$.

Another question posed in [3] is to determine the minimum number of *vertices* a 3-uniform hypergraph having Property O can have.

Definition 1.4. For $k \geq 2$ we define

$$n(k) := \min\{|V| : \text{there exists an oriented } k\text{-graph } \mathcal{H} = (V, \mathcal{E}) \text{ having Property O}\}$$

Duffus et al. proved that $6 \leq n(3) \leq 9$. For the upper bound they gave a construction and the lower bound was proved using an exhaustive computer search. In Section 3 we prove that $n(3) = 6$ by providing two different constructions.

2 Proof of Theorem 1.3

To motivate the construction, we will first consider the case $k = 3$.

Claim 2.1. *There exists an oriented 3-graph with 10 edges having Property O, i.e. $f(3) \leq 10$. Furthermore $f(3) \geq 7$.*

The lower bound $f(3) \geq 7$ was already mentioned in [3]. We will include its simple proof for the convenience of the reader. It is worth noting that this is the only non-trivial case in which a lower bound of the form $k! + 1$ is known.

Before proving this claim, let us describe the idea of the proof of the upper bound. We start by defining two edges (x, y, a) and (y, x, b) . Any ordering is consistent with the relative order of exactly one of these edges with respect to the positions of x and y . If it happens to be $x < y$, but the edge (x, y, a) is not consistent with the ordering, then there are two possibilities for the position of a with respect to both x and y . For each possibility we introduce one new vertex and two edges, such that at least one of them is consistent with the ordering. Below are the details. To simplify the notation of the generalization, we used more vertices than actually needed. Indeed the number of vertices in the following construction can be reduced (see Section 3).

Proof of Claim 2.1. We will first show that $f(3) \leq 10$. To do so, let \mathcal{H} be an oriented 3-graph with vertex set $V = \{x, y, a, b, c, d, e, f\}$, and edge set

$$\mathcal{E} = \{(x, y, a), (a, x, c), (c, x, y), (x, a, d), (d, a, y), (y, x, b), (b, y, e), (e, y, x), (y, b, f), (f, b, x)\}$$

We have to show that \mathcal{H} has Property O. Let $<$ be an arbitrary ordering of V . Since either $x < y$ or $y < x$ at most one of the edges (x, y, a) and (y, x, b) can be consistent with $<$. Let us first assume $x < y$. If (x, y, a) is not consistent with $<$, then we either have $a < x < y$ or $x < a < y$. If $a < x < y$, then at least one of (a, x, c) or (c, x, y) is $<$ -consistent. On the other hand, if $x < a < y$ then at least one of (x, a, d) or (d, a, y) is $<$ -consistent. Now, if $y < x$ but (y, x, b) is not $<$ -consistent, then either $b < y < x$ or $y < b < x$. In the first case at least one of (b, y, e) or (e, y, x) is $<$ -consistent and in the latter at least one of (y, b, f) or (f, b, x) is $<$ -consistent. Hence \mathcal{H} has Property O, proving $f(3) \leq 10$.

We will now prove the lower bound. Suppose for a contradiction that $\mathcal{H} = (V, \mathcal{E})$ is 3-graph with 6 edges that has Property O. As already mentioned, every edge is consistent with $\frac{n!}{3!}$ linear orders and so $|\mathcal{E}| \cdot \frac{n!}{3!} = n!$. This means that no collection of two edges is consistent with with same linear order. This forces that every pair of edges of \mathcal{H} intersects in exactly two vertices: if there were a pair of edges $\overline{e_1}, \overline{e_2}$ with $|e_1 \cap e_2| \leq 1$, then clearly these two edges are consistent with at least 2 linear orders. As we have at least 6 vertices, this means that \mathcal{H} is a sunflower with core x, y , say, and at least 3 pedals. Therefore there is a pair of edges for which the relative order of x, y is equal. It is not hard to see that these two edges are consistent with at least 2 linear orders giving the desired contradiction. Hence $f(3) \geq 7 = 3! + 1$. \square

Before proving Theorem 1.3 we will introduce some notation. Given a finite set V , set $\mathbf{x} := \{x_1, \dots, x_{k-1}\} \in V^{k-1}$. Enumerate all permutations of $[k-1]$ by $\pi_1 = \text{id}, \dots, \pi_{(k-1)!}$ arbitrarily. Furthermore, for $j \in [(k-1)!]$ and $i \in [k-1]$, write $\pi_j(\mathbf{x}, i(y))$ to denote the k -tuple arising from the $(k-1)$ -tuple $\pi_j(\mathbf{x})$ by putting y between $\pi_j(x_{i-1})$ and $\pi_j(x_i)$ (when we write “between $\pi_j(x_0)$ and $\pi_j(x_1)$ ”, we mean “before $\pi_j(x_1)$ ”).

The construction is a fairly straightforward generalization of the $k = 3$ construction. We start with $(k-1)!$ edges $(\pi_1(\mathbf{x}), a_1), \dots, (\pi_{(k-1)!}(\mathbf{x}), a_{(k-1)!})$. Any permutation will be consistent with the relative order of the x_i for exactly one of the above k -tuples. Say it is with $\pi_1 = \text{id}$ (in any other case we proceed in the same way), but the k -tuple $(\pi_1(\mathbf{x}), a_1)$ is not π_1 -consistent. Then there are $(k-1)$ places for a_1 and for each possibility we introduce one new vertex and $\lfloor \frac{k}{2} \rfloor + 1$ edges such that at least one of them will be π_1 -consistent.

Proof of Theorem 1.3. We begin with the construction of the desired hypergraph $\mathcal{H} = (V, \mathcal{E})$. Let the set of vertices be

$$V = \left\{ x_1, x_2, \dots, x_{k-1}, a_1, \dots, a_{(k-1)!}, a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(k-1)}, \dots, a_{(k-1)!}^{(1)}, a_{(k-1)!}^{(2)}, \dots, a_{(k-1)!}^{(k-1)} \right\}.$$

and let \mathcal{E} be the set of ordered hyperedges (k -tuples) constructed as follows (compare with the example given below):

- (1) Start with the $(k-1)!$ k -tuples of the form $(\pi_j(\mathbf{x}), a_j)$, for $j = 1, 2, \dots, (k-1)!$, and put them into \mathcal{E} .
- (2) For each fixed $j \in [(k-1)!]$ put the following $(k-1) \left(\lfloor \frac{k}{2} \rfloor + 1 \right)$ k -tuples into \mathcal{E} : For every $i \in [k-1]$, consider the k -tuple $\pi_j(\mathbf{x}, i(a_j))$.

Now, for every odd position $l \in [k-1]$, replace the l -th element of $\pi_j(\mathbf{x}, i(a_j))$ by $a_i^{(j)}$ and put it into \mathcal{E} . Also, replace the k -th element with $a_i^{(j)}$ and put the resulting k -tuple into \mathcal{E} .

To prove that \mathcal{H} has Property O, one proceeds precisely as in the proof of Claim 2.1. To finish the proof of Theorem 1.3, note that, using the paragraph just above the proof, we have

$$|\mathcal{E}| = ((k-1) \left(\lfloor \frac{k}{2} \rfloor + 1 \right) + 1) (k-1)! = \left(\lfloor \frac{k}{2} \rfloor + 1 \right) k! - \lfloor \frac{k}{2} \rfloor (k-1)!,$$

completing the proof. \square

Example: Let us illustrate the construction in the proof of Theorem 1.3 for $k = 4$. We start with the edges $(\pi_j(\mathbf{x}), a_j)$ for $j = 1, 2, \dots, 6$. Now suppose $j = 1$, i.e. $\pi_j = \text{id}$ and define the following edges:

$i = 1$ we have (a_1, x_1, x_2, x_3) and so we put

$$(a_1^{(1)}, x_1, x_2, x_3) \text{ and } (a_1, x_1, a_1^{(1)}, x_3) \text{ and } (a_1, x_1, x_2, a_1^{(1)}) \text{ into } \mathcal{E};$$

$i = 2$ we have (x_1, a_1, x_2, x_3) and so we put

$$(a_2^{(1)}, a_1, x_2, x_3) \text{ and } (x_1, a_1, a_2^{(1)}, x_3) \text{ and } (x_1, a_1, x_2, a_2^{(1)}) \text{ into } \mathcal{E};$$

$i = 3$ we have (x_1, x_2, a_1, x_3) and so we put

$$(a_3^{(1)}, x_2, a_1, x_3) \text{ and } (x_1, x_2, a_3^{(1)}, x_3) \text{ and } (x_1, x_2, a_1, a_3^{(1)}) \text{ into } \mathcal{E}.$$

For $j = 2, \dots, 6$ one proceeds similarly.

Remark 2.2. Note that the number of vertices in the construction above is $k! + k - 1$.

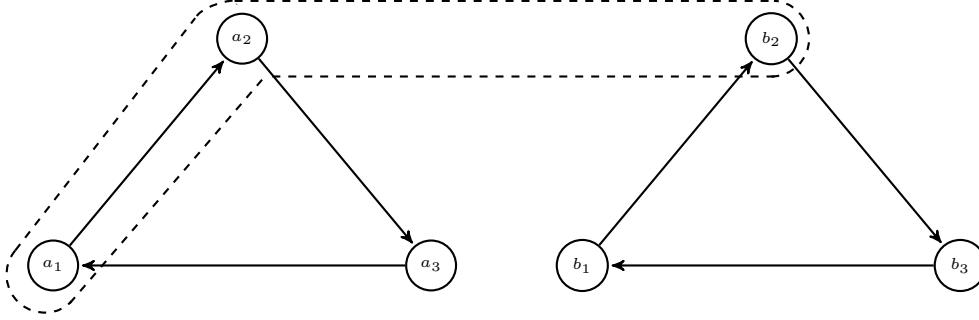


Figure 1: The 3-graph \mathcal{H} with the edge (a_1, a_2, b_2) depicted.

3 3-uniform hypergraphs on 6 vertices having property O

In this short section we will construct two oriented 3-uniform hypergraphs on 6 vertices having Property O. Combined with the lower bound given in [3], this shows $n(3) = 6$.

The first one is obtained from a graph having Property O. Firstly, recall that a cyclicly ordered triangle is a graph having Property O.

Now take two disjoint copies of an oriented triangle, say $(a_0, a_1), (a_1, a_2), (a_2, a_0)$ and $(b_0, b_1), (b_1, b_2), (b_2, b_0)$ and define the following oriented edge set (where we take the indices mod 3):

$$\mathcal{E}_0 = \{(a_i, a_{i+1}, b_j), (b_i, b_{i+1}, a_j) : 0 \leq i, j \leq 2\}.$$

Claim 3.1. *The oriented 3-graph \mathcal{H} with vertex set $V = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ and edge set \mathcal{E}_0 has Property O. Thus $n(3) \leq 6$.*

Proof. Let π be an arbitrary ordering of V . We have to show that there is an edge $\bar{e} \in \mathcal{E}_0$ that is π -consistent.

Now, if there is some b_i such that b_i is greater than every a_j (with respect to π), then, since the 2-graph induced by the vertices a_0, a_1, a_2 has Property O, some edge of the form (a_j, a_{j+1}, b_i) is π -consistent.

If not, then there exists an a_i that is greater than every b_j . So by symmetry, the same argument as above shows that there exists an edge of the form (b_j, b_{j+1}, a_i) that is π -consistent. Hence $n(3) \leq 6$. \square

The second example is a simple modification of the construction given in Section 2. Indeed, instead of using the vertices e and f , we could have used c and d again: Simply replace e by d and f by c . So we get $V = \{x, y, a, b, c, d\}$ and

$$\mathcal{E} = \{(x, y, a), (a, x, c), (c, x, y), (x, a, d), (d, a, y), (y, x, b), (b, y, d), (d, y, x), (y, b, c), (c, b, x)\}$$

One can use the same proof to show that this oriented 3-graph has Property O. Similar modifications can be made in the construction in Theorem 1.3 to lower the number of vertices. However this is slightly more tedious and not the aim of the construction.

4 Concluding Remarks

In this note we showed $f(k) \leq (\lfloor \frac{k}{2} \rfloor + 1) k! - \lfloor \frac{k}{2} \rfloor (k-1)!$ for every $k \geq 3$. The main problem regarding hypergraphs having Property O is the following:

Problem 4.1. *Is it true that $\frac{f(k)}{k!} \rightarrow \infty$ as $k \rightarrow \infty$?*

In Section 2 we saw that $f(3) \geq 3! + 1$. Improving the lower bound seems to be the main task. A first step would be to answer the following question.

Problem 4.2. *Is it true that $f(k) \geq k! + 1$ for every $k \geq 3$?*

We believe that the answer should be yes. Of course, an improvement of the upper bound would be interesting as well.

Problem 4.3. *Let $k \geq 3$. Is there a k -uniform hypergraph with $n(k)$ vertices and $f(k)$ edges having Property O?*

The second construction in Section 3 has fairly few edges, namely 10, and $n(3) = 6$ vertices.

Note that a trivial lower bound on $n(k)$ is $(\frac{k}{e})^2$ (for every $k \geq 2$), since the number of edges is larger than $k!$ and smaller than $\binom{n}{k}$. On the other hand, Duffus et al. [3, pp. 3–4] showed that a k -tournament $\mathcal{T}_{n,k}$ on $n = (\frac{k}{e})^2 (\pi \cdot \exp(e^2/2) \cdot k^3 \ln k)^{1/k}$ vertices with all the $\binom{n}{k}$ edges ordered randomly has Property O with positive probability. Hence

$$n(k) \leq \left(\frac{k}{e}\right)^2 (\pi \cdot \exp(e^2/2) \cdot k^3 \ln k)^{1/k} = (1 + o(1)) \left(\frac{k}{e}\right)^2,$$

and so $n(k) = (1 + o(1)) \left(\frac{k}{e}\right)^2$.

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